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A connection between twistors and superstring sigma models on coset superspaces

Martin Wolf[1]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.

E-mail: m.wolf@damtp.cam.ac.uk

Abstract: We consider superstring sigma models that are based on coset superspaces G/H in which H arises as the fixed point set of an order-4 automorphism of G. We show by means of twistor theory that the corresponding first-order system, consisting of the Maurer-Cartan equations and the equations of motion, arises from a dimensional reduction of some generalised self-dual Yang-Mills equations in eight dimensions. Such a relationship might help shed light on the explicit construction of solutions to the superstring equations including their hidden symmetry structures and thus on the properties of their gauge theory duals.

Keywords: Integrable Equations in Physics, Superstrings and Heterotic Strings, AdS-CFT Correspondence, Integrable Field Theories

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[1] Also at the Wolfson College, Barton Road, Cambridge CB3 9BB, U.K.

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1 Introduction

Remarkable advancements in our understanding of maximally $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory have been made possible due to its integrability in the planar limit. This theory appears to be equivalent to type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ via the AdS/CFT correspondence and in particular at strong coupling it is described by classical superstrings. In [1], the well-known classical integrability of the bosonic $\mathrm{AdS}_5 \times S^5$ string sigma model was shown to extend to its κ-symmetric Green-Schwarz-type fermionic generalisation [2] (see also [3]). In this formulation, the superstring sigma model action is based on the coset superspace $\mathsf{PSU}(2,2|4)/(\mathsf{SO}(1,4) \times \mathsf{SO}(5))$, where the denominator group arises as fixed point set of an order-4 automorphism of $\mathsf{PSU}(2,2|4)$. It is this latter feature that allows for the construction of conserved non-local charges à la Lüscher and Pohlmeyer [4] for the superstring on $\mathrm{AdS}_5 \times S^5$ [1] (see also [5–9]).[1] For a discussion of integrability of the superstring model with the gauges fixed and the Virasoro constraints imposed, see [11, 12] and e.g. [13–24].

In this work, we consider superstring sigma models that are based on coset superspaces G/H. Even though our analysis can be extended to more general cases, we always assume that H arises as fixed point set of an order-4 automorphism. This then includes the above-mentioned case of type IIB superstrings on $\mathrm{AdS}_5 \times S^5$. This also includes type IIA superstring theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ in a peculiar partial κ-symmetry gauge with $G/H = \mathsf{OSp}(2,2|6)/(\mathsf{SO}(1,3) \times \mathsf{U}(3))$ [25, 26] (see also [27, 28]). This particular string theory is the gravitational dual of the 't Hooft limit of a three-dimensional Chern-Simons matter theory that has recently been proposed to be the low-energy description of stacks of M2-branes on $\mathbb{R}^8/\mathbb{Z}_k$ [29]. Notice that it is a quite generic feature that superstring sigma models

[1] Aspects related to involutivity of the charges were discussed in [10].

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based on coset superspaces of the above type are classically integrable and in fact, this even extends to models on coset (super)spaces with order-k automorphisms [30].

The full system of the corresponding superstring equations consists of i) the Maurer-Cartan equations and ii) the equations of motion that follow upon varying an associated action functional. This set of equations is referred to as the first-order system for the superstring.[2] We shall discuss the integrability of this system from a different point of view: By using twistor methods, we show that the first-order system of the superstring arises via a dimensional reduction of some generalised self-dual Yang-Mills (SDYM) theory in eight dimensions. The reason for considering eight dimensions lies in the necessity of having three 'Higgs fields' (as a result of the $\mathbb{Z}_4$-grading) after the dimensional reduction. Recall that there are various generalisations of the four-dimensional SDYM equations to $\mathbb{R}^d$ with $d > 4$ [31–33] and some solutions to these generalised equations were, for instance, constructed in [33–39]. See also [40] for an extension of the ADHM construction [41]. Below, we will identify the theory that gives rise to the Lax formulation of superstring theory on G/H. Before discussing the superstring case, however, we shall review the case of symmetric space sigma models thereby setting up our notation and conventions.

Since the present approach is based on twistor theory, one may naturally hope that it will turn out useful for the construction of explicit solutions to the superstring equations of motion by e.g. using twistor methods like Ward's splitting approach [42] (see also [32]) and for the study of the hidden symmetry structures. This in turn would shed light on the properties of (strongly coupled) gauge theory via the holographic correspondence. We will briefly comment on this at the end of this work.

2 Symmetric space coset models and self-dual Yang-Mills theory

2.1 Symmetric space coset models

Let G be a Lie group and H a Lie subgroup of G and consider the coset $G/H := \{gH \,|\, g \in G\}$. We shall assume that H arises as the fixed point set of an order-2 automorphism of G. This means that at the Lie algebra level $\mathfrak{g} := \mathrm{Lie}(G)$ we have a $\mathbb{Z}_2$-decomposition according to $\mathfrak{g} \cong \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(2)}$, where $\mathfrak{g}_{(0)} := \mathrm{Lie}(H)$ and

$$[\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}] \subset \mathfrak{g}_{(0)}\,, \quad [\mathfrak{g}_{(0)}, \mathfrak{g}_{(2)}] \subset \mathfrak{g}_{(2)} \quad \text{and} \quad [\mathfrak{g}_{(2)}, \mathfrak{g}_{(2)}] \subset \mathfrak{g}_{(0)}\,. \tag{2.1}$$

If these relations are satisfied, G/H is said to be a symmetric space. In the sequel, we shall also denote $\mathfrak{g}_{(0)}$ by $\mathfrak{h}$.

To define the sigma model action, we consider a map $g : \Sigma \to G$, where Σ is a world-sheet surface with a metric of Lorentzian signature $(+-)$, and introduce the flat current

$$j := g^{-1}\mathrm{d}g = j_{(0)} + j_{(2)} = A + j_{(2)}\,, \quad \text{with} \quad A := j_{(0)} \in \mathfrak{h} \quad \text{and} \quad j_{(2)} \in \mathfrak{g}_{(2)}\,. \tag{2.2}$$

The dynamical two-dimensional fields will take values in the coset space G/H. The action that describes them should simultaneously be invariant under the global (left) G-transformations of the form

$$g \mapsto g_0 g \quad \text{for} \quad g_0 \in G\,, \tag{2.3a}$$

[2] Of course, these equations should be complemented by the Virasoro constraints.

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and the local (right) H-transformations of the form

$$g \;\mapsto\; gh \quad \text{for} \quad h \;\in\; H\;. \tag{2.3b}$$

By construction, the current j is invariant under (2.3a). Under (2.3b), the A-part of j in (2.2) transforms as a connection, $A \mapsto h^{-1}Ah+h^{-1}\mathrm{d}h$, while $j_{(0)}$ transforms covariantly, $j_{(0)} \mapsto h^{-1}j_{(0)}h$.

The sigma model action is then given by

$$S \;=\; \tfrac{1}{2}\int_\Sigma \operatorname{tr}\left[j_{(2)}\wedge *j_{(2)}\right]\;. \tag{2.4}$$

Here, '$*$' is the Hodge star operator on Σ and 'tr' the trace on $\mathfrak{g}$ compatible with the $\mathbb{Z}_2$-grading. If we set

$$\nabla\alpha \;:=\; \mathrm{d}\alpha + A\wedge\alpha - (-1)^p\alpha\wedge A \tag{2.5}$$

for a Lie algebra-valued p-form α on Σ, then the corresponding first-order system may be written as

$$\mathrm{d}A + A\wedge A + j_{(2)}\wedge j_{(2)} \;=\; 0\,,\quad \nabla j_{(2)} \;=\; 0 \quad\text{and}\quad \nabla{*j_{(2)}} \;=\; 0\,, \tag{2.6}$$

where the first two equations are the $\mathfrak{h}$ and $\mathfrak{g}_{(2)}$ components of the Maurer-Cartan equation. As is well-known, the first-order system is equivalent to the flatness,

$$\mathrm{d}J(\zeta) + J(\zeta)\wedge J(\zeta) \;=\; 0\,, \tag{2.7a}$$

of a Lax connection $J(\zeta)$, with ζ a complex spectral parameter:

$$J(\zeta) \;:=\; A + \tfrac{1}{2}(\zeta+\zeta^{-1})\,j_{(2)} + \tfrac{1}{2}(\zeta-\zeta^{-1})\,{*j_{(2)}}\;. \tag{2.7b}$$

To arrive at (2.6) from (2.7) we note that on a worldsheet Σ with a Lorentzian signature metric we have $**=1$. We also have $\alpha\wedge *\beta + *\alpha\wedge\beta = 0$ for two one-forms α and β on Σ. Notice that the flatness equation (2.7a) follows as compatibility condition for an auxiliary linear problem

$$\left[\mathrm{d}+J(\zeta)\right]\psi \;=\; 0\,, \tag{2.7c}$$

where ψ is some G-valued function that depends on the spectral parameter ζ.

2.2 Twistors and self-dual Yang-Mills theory

Let us now explain how the system (2.6), (2.7) in conformal gauge arises from SDYM theory in four dimensions. To this end, we start from the twistor approach. For text-book treatments of SDYM theory in the context of twistor theory, we refer to [43, 44].

Consider complexified four-dimensional space-time $\mathcal{M}^4 := \mathbb{C}^4$. We have the identification $T\mathcal{M}^4 \cong \mathcal{S}\otimes\tilde{\mathcal{S}}$, where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are the two spinor bundles of undotted and dotted spinors on $\mathcal{M}^4$, and so we may consider the projective co-spin bundle $\mathcal{F}^5 := \mathbb{P}(\tilde{\mathcal{S}}^*) \cong \mathbb{C}^4\times\mathbb{C}P^1$ over $\mathcal{M}^4$. We shall refer to $\mathcal{F}^5$ as correspondence space. The spaces $\mathcal{M}^4$ and $\mathcal{F}^5$ may be coordinatised by $x^{\alpha\dot\beta}$ and $(x^{\alpha\dot\beta},\lambda_{\dot\alpha})$, where $\lambda_{\dot\alpha}$ are homogeneous coordinates on $\mathbb{C}P^1$ and $\alpha,\beta,\ldots = 1,2,\ \dot\alpha,\dot\beta,\ldots = \dot1,\dot2$. On the spinor space $\mathcal{S}$ (and similarly on $\tilde{\mathcal{S}}$) we have a

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symplectic form $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$ with $\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta$ and $\varepsilon_{12} = -1$, which can be used to raise and lower spinor indices. If we let $\partial_{\alpha\dot\beta} := \partial/\partial x^{\alpha\dot\beta}$, then we define the twistor distribution to be the rank-2 distribution $\mathcal{D}$ on $\mathcal{F}^5$ given by

$$\mathcal{D} \;:=\; \mathrm{span}\left\{V_\alpha := \lambda^{\dot\beta}\partial_{\alpha\dot\beta}\right\} . \tag{2.8}$$

Since $\mathcal{D}$ is integrable, it defines a foliation of $\mathcal{F}^5$, the resulting quotient will be twistor space, a three-dimensional complex manifold denoted by $\mathcal{P}^3$. We have thus established the following double fibration:

$$\begin{tikzcd} & \mathcal{F}^5 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{P}^3 & & \mathcal{M}^4 \end{tikzcd} \tag{2.9}$$

where π_2 is the trivial projection and $\pi_1 : (x^{\alpha\dot\beta}, \lambda_{\dot\alpha}) \mapsto (z^\alpha = x^{\alpha\dot\beta}\lambda_{\dot\beta}, \lambda_{\dot\alpha})$. Hence, $\mathcal{P}^3 \subset \mathbb{C}P^3$ can be identified with $\mathcal{O}(1) \otimes \mathbb{C}^2 \to \mathbb{C}P^1$, where $\mathcal{O}(m)$ are the homogeneous polynomials of degree m on $\mathbb{C}P^1$. Furthermore, a point $x \in \mathcal{M}^4$ corresponds to a projective line $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ in twistor space, while a point $(z, \lambda) \in \mathcal{P}^3$ corresponds to a totally two-dimensional null-plane in space-time $\mathcal{M}^4$. Such a plane may be parametrised as $x^{\alpha\dot\beta} = x_0^{\alpha\dot\beta} + \mu^\alpha\lambda^{\dot\beta}$, with $x_0^{\alpha\dot\beta} = \text{const.}$ and μ^α arbitrary.

Consider now a rank-r holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^3$ and its pull-back $\pi_1^*\mathcal{E} \to \mathcal{F}^5$.[3] Both the twistor space and the correspondence space can be covered by two coordinate patches which we denote by $\mathcal{U}_\pm$ and $\hat{\mathcal{U}}_\pm$, respectively. Then the bundles $\mathcal{E}$ and $\pi_1^*\mathcal{E}$ are characterised by transition functions f_{+-} on $\mathcal{U}_+ \cap \mathcal{U}_-$ and $\pi_1^* f_{+-}$ on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. In the sequel, we shall not make a notational distinction between f_{+-} and $\pi_1^* f_{+-}$ and simply write f_{+-} for both bundles. By definition of a pull back, f_{+-} is constant along $\pi_1 : \mathcal{F}^5 \to \mathcal{P}^3$ and thus is annihilated by the vector fields of the twistor distribution (2.8). Letting $\bar\partial_{\mathcal{P}}$ and $\bar\partial_{\mathcal{F}}$ be the anti-holomorphic parts of the exterior derivatives on $\mathcal{P}^3$ and $\mathcal{F}^5$, respectively, we have $\pi_1^*\bar\partial_{\mathcal{P}} = \bar\partial_{\mathcal{F}} \circ \pi_1^*$. Hence, the transition function f_{+-} is also annihilated by $\bar\partial_{\mathcal{F}}$.

We shall also assume that $\mathcal{E}$ is topologically trivial and holomorphically trivial when restricted to any $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ for $x \in \mathcal{M}^4$. These conditions then imply the existence of smooth $\mathsf{GL}(r, \mathbb{C})$-valued functions $\psi_\pm$ on $\hat{\mathcal{U}}_\pm$ such that f_{+-} can be decomposed as $f_{+-} = \psi_+^{-1}\psi_-$ with $\bar\partial_{\mathcal{F}}\psi_\pm = 0$, i.e. the $\psi_\pm$ are holomorphic on $\hat{\mathcal{U}}_\pm$. Clearly, this splitting is not unique, since one can always perform the transformation $\psi_\pm \mapsto g\psi_\pm$, where g is some globally defined $\mathsf{GL}(r, \mathbb{C})$-valued holomorphic function on $\mathcal{F}^5$ (hence it is constant on $\mathbb{C}P^1$). The choice of g will correspond to a choice of gauge for the Yang-Mills gauge potential on space-time. Since $V_\alpha^\pm f_{+-} = 0$, where $V_\alpha^\pm$ are the restrictions of V_α to the coordinate patches $\hat{\mathcal{U}}_\pm$, we find

$$\psi_+ V_\alpha^+ \psi_+^{-1} \;=\; \psi_- V_\alpha^+ \psi_-^{-1} \tag{2.10}$$

on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. Explicitly, $V_\alpha^\pm = \lambda_\pm^{\dot\beta}\partial_{\alpha\dot\beta}$ with $\lambda_{\dot\alpha}^+ := \lambda_{\dot\alpha}/\lambda_{\dot 1} =: (1, \lambda_+)^T$ and $\lambda_{\dot\alpha}^- := \lambda_{\dot\alpha}/\lambda_{\dot 2} =: (\lambda_-, 1)^T$, where $(x^{\alpha\dot\beta}, \lambda_\pm)$ are local coordinates on $\hat{\mathcal{U}}_\pm$. Therefore, by an extension of

[3] One may impose the additional condition of having a trivial determinant line bundle $\det \mathcal{E}$ what would reduce the structure group $\mathsf{GL}(r, \mathbb{C})$ to $\mathsf{SL}(r, \mathbb{C})$.

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Liouville's theorem, the expressions (2.10) can be at most linear in λ_+ and thus we may introduce a Lie algebra-valued one-form $\mathcal{A}$ on $\mathcal{F}^5$ which has components only along $\mathcal{D}$,

$$V_\alpha \lrcorner \mathcal{A}|_{\hat{\mathcal{U}}_\pm} \;:=\; \mathcal{A}^\pm_\alpha \;=\; \psi_\pm V^\pm_\alpha \psi^{-1}_\pm \;=\; \lambda^{\dot\beta}_\pm \mathcal{A}_{\alpha\dot\beta}\,, \tag{2.11}$$

where $\mathcal{A}_{\alpha\dot\beta}$ is $\lambda_\pm$-independent. This can be re-written as

$$(V^\pm_\alpha + \mathcal{A}^\pm_\alpha)\psi_\pm \;=\; \lambda^{\dot\beta}_\pm \nabla_{\alpha\dot\beta}\psi_\pm \;=\; 0\,, \quad \text{with} \quad \nabla_{\alpha\dot\beta} \;:=\; \partial_{\alpha\dot\beta} + \mathcal{A}_{\alpha\dot\beta}\,. \tag{2.12}$$

The compatibility conditions for this linear system read as

$$[\nabla_{\alpha\dot\beta}, \nabla_{\gamma\dot\delta}] + [\nabla_{\alpha\dot\delta}, \nabla_{\gamma\dot\beta}] \;=\; 0\,, \tag{2.13}$$

and this is nothing but the SDYM equations, since

$$[\nabla_{\alpha\dot\beta}, \nabla_{\gamma\dot\delta}] \;=\; \varepsilon_{\alpha\gamma} f_{\dot\beta\dot\delta} + \varepsilon_{\dot\beta\dot\delta} f_{\alpha\beta}\,, \tag{2.14}$$

where $f_{\alpha\beta}$ (respectively, $f_{\dot\alpha\dot\beta}$) represents the self-dual (respectively, anti-self-dual) part of the field strength.

In summary, we have described a one-to-one correspondence between equivalence classes of holomorphic vector bundles[4] over twistor space that are holomorphically trivial on any projective live $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ and gauge equivalence classes of solutions to the SDYM equations on $\mathcal{M}^4$. This is called the Penrose-Ward transform [42, 45].

Let us now introduce a real structure on $\mathcal{P}^3$ that yields a split signature real slice in $\mathcal{M}^4$. This can be done by introducing an anti-holomorphic involution $\tau\;:\;\mathcal{P}^3 \to \mathcal{P}^3$ that is given by

$$\tau(z^\alpha, \lambda_{\dot\alpha}) \;:=\; (\bar z^\beta C_\beta{}^\alpha, C_{\dot\alpha}{}^{\dot\beta}\bar\lambda_{\dot\beta})\,, \tag{2.15a}$$

where bar denotes complex conjugation and[5]

$$(C_\alpha{}^\beta) \;:=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (C_{\dot\alpha}{}^{\dot\beta}) \;:=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.15b}$$

By virtue of the incidence relation $z^\alpha = x^{\alpha\dot\beta}\lambda_{\dot\beta}$, we obtain an induced involution on $\mathcal{M}^4$,[6]

$$\tau(x^{\alpha\dot\beta}) \;=\; \bar x^{\gamma\dot\delta} C_\gamma{}^\alpha C_{\dot\delta}{}^{\dot\beta}\,. \tag{2.16}$$

The fixed point set $\tau(x) = x$ is given by $x^{1\dot1} = \bar x^{2\dot2}$ and $x^{1\dot2} = \bar x^{1\dot2}$ and defines a split signature space-time $\mathcal{M}^4_\tau \cong \mathbb{R}^{2,2}$:

$$\mathrm{d}s^2 \;=\; -\frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot\gamma\dot\delta}\mathrm{d}x^{\alpha\dot\gamma}\mathrm{d}x^{\beta\dot\delta} \;=\; -|\mathrm{d}x^{1\dot1}|^2 + |\mathrm{d}x^{1\dot2}|^2\,. \tag{2.17}$$

[4]Recall that the holomorphic vector bundles $\mathcal{E}$ and $\pi_1^*\mathcal{E}$ are defined up to the equivalence $f_{+-} \sim h_+^{-1} f_{+-} h_-$, where the $h_\pm$ are holomorphic $\mathsf{GL}(r,\mathbb{C})$-valued function on $\hat{\mathcal{U}}_\pm$ and $V^\pm_\alpha h_\pm = 0$.

[5]Note that these are nothing but the charge conjugation matrices in split signature [46]. Such changes do not affect $\mathcal{A}_{\alpha\dot\beta}$.

[6]We shall use the same notation τ for the anti-holomorphic involutions induced on the different manifolds appearing in (2.9).

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We may choose the following parametrisation:

$$x^{1\dot{1}} = \bar{x}^{2\dot{2}} =: -(x^3 - \mathrm{i}x^2) \quad \text{and} \quad x^{1\dot{2}} = \bar{x}^{2\dot{1}} =: (x^4 + \mathrm{i}x^1) \tag{2.18}$$

leading to

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2 \,. \tag{2.19}$$

Notice that the involution τ can be extended to (holomorphic) functions defined on the manifolds appearing in the double fibration (2.9) and hence to $\mathcal{E}$ and $\pi_1^*\mathcal{E}$ yielding real gauge fields taking values in some real form $\mathfrak{g}$ of $\mathfrak{gl}(r,\mathbb{C})$. For a detailed account on the real geometries appearing for (2.15), we refer to [47].

Using the coordinates $x^\mu = (x^1,\ldots,x^4)$ and $\partial_\mu := \partial/\partial x^\mu$, the SDYM equations (2.13) take the more familiar form

$$\mathcal{F}_{12} = -\mathcal{F}_{34}\,, \quad \mathcal{F}_{13} = \mathcal{F}_{24} \quad \text{and} \quad \mathcal{F}_{14} = \mathcal{F}_{23}\,, \tag{2.20}$$

with $\mathcal{F}_{\mu\nu} := [\nabla_\mu, \nabla_\nu]$ and $\nabla_\mu := \partial_\mu + \mathcal{A}_\mu$, while the linear system on e.g. $\hat{\mathcal{U}}_+$ is given by

$$\begin{gathered} \mathcal{L}_1\psi = 0 = \mathcal{L}_2\psi\,, \\ \mathcal{L}_1 := \lambda(\nabla_3 + \mathrm{i}\nabla_2) + (\nabla_4 - \mathrm{i}\nabla_1) \quad \text{and} \quad \mathcal{L}_2 := \lambda(\nabla_4 + \mathrm{i}\nabla_1) + (\nabla_3 - \mathrm{i}\nabla_2)\,, \end{gathered} \tag{2.21}$$

with $\lambda := \lambda_+$ and $\psi := \psi_+$.

To make contact with the discussion of the previous section, let us perform the linear fractional transformation

$$\lambda = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}} \tag{2.22}$$

upon which the linear system (2.21) becomes

$$\begin{gathered} \hat{\mathcal{L}}_1\psi = 0 = \hat{\mathcal{L}}_2\psi\,, \\ \hat{\mathcal{L}}_1 := \zeta(\nabla_3 + \nabla_4) + (\nabla_1 + \nabla_2) \quad \text{and} \quad \hat{\mathcal{L}}_2 := \zeta(\nabla_1 - \nabla_2) + (\nabla_3 - \nabla_4)\,. \end{gathered} \tag{2.23}$$

Of course, also this linear system leads to (2.20). Assuming that the gauge potential $\mathcal{A}_\mu$ depends only on x^1 and x^2 together with $\Phi_{1,2} := \mathcal{A}_{3,4}$ and taking the linear combinations $\frac{1}{2}[\hat{\mathcal{L}}_1 \pm \zeta^{-1}\hat{\mathcal{L}}_2]$, we find

$$\begin{aligned} \left[\partial_1 + \mathcal{A}_1 + \frac{1}{2}(\zeta + \zeta^{-1})\Phi_1 + \frac{1}{2}(\zeta - \zeta^{-1})\Phi_2\right]\psi &= 0\,, \\ \left[\partial_2 + \mathcal{A}_2 + \frac{1}{2}(\zeta + \zeta^{-1})\Phi_2 + \frac{1}{2}(\zeta - \zeta^{-1})\Phi_1\right]\psi &= 0\,. \end{aligned} \tag{2.24a}$$

Thus, we arrive at

$$\mathcal{F}_{12} + [\Phi_1, \Phi_2] = 0\,, \quad \nabla_1\Phi_2 - \nabla_2\Phi_1 = 0 \quad \text{and} \quad \nabla_1\Phi_1 - \nabla_2\Phi_2 = 0\,. \tag{2.24b}$$

Let us write $\mathcal{A} = \mathcal{A}_1\mathrm{d}x^1 + \mathcal{A}_2\mathrm{d}x^2$ and $\Phi = \Phi_1\mathrm{d}x^1 + \Phi_2\mathrm{d}x^2$. The system (2.24) is almost the component form of (2.6), (2.7) when written in conformal gauge. In fact, assuming that $\mathfrak{g}$ admits a $\mathbb{Z}_2$-grading as discussed above, the sigma model equations on G/H arise as a $\mathbb{Z}_2$-invariant subsector of (2.24) determined by this grading. If we let $\Omega : \mathfrak{g} \to \mathfrak{g}$ be

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the $\mathbb{Z}_2$-automorphism of $\mathfrak{g}$, we may introduce the projectors $\mathscr{P}_{(0)} := \frac{1}{2}(1+\Omega)$ and $\mathscr{P}_{(2)} := \frac{1}{2}(1-\Omega)$ such that $\mathfrak{h} = \mathscr{P}_{(0)}(\mathfrak{g})$ and $\mathfrak{g}_{(2)} = \mathscr{P}_{(2)}(\mathfrak{g})$. The configurations $(\mathcal{A}, \Phi)$ we are interested in are then those with $(\mathcal{A}, \Phi) = (\Omega(\mathcal{A}), -\Omega(\Phi))$, i.e. we may set $\mathcal{A} =: A \in \mathfrak{h}$ and $\Phi =: j_{(2)} \in \mathfrak{g}_{(2)}$. Notice that the system (2.24) was introduced in [48, 49] (see also [50, 51]). Notice also that one may study dimensional reductions of the $\mathcal{N} = 4$ SDYM equations on $\mathbb{R}^{2,2}$ to two dimensions as done in [52] to end up with sigma models for maps from certain super Riemann surfaces into G or G/H.

In summary, the SDYM equations (2.20) for a gauge potential $\mathcal{A}_\mu$ on four-dimensional flat space with a split signature metric and with $\mathcal{A}_{1,2} \in \mathfrak{h}$ and $\mathcal{A}_{3,4} \in \mathfrak{g}_{(2)}$ for $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}_{(2)}$ will reduce to the first-order system of a coset model G/H in conformal gauge, provided we assume that $\mathcal{A}_\mu$ is independent of x^3 and x^4.

3 Superstring sigma models and generalised self-dual Yang-Mills theory

3.1 Superstring sigma models

Let us now examine superstring models that are based on coset superspaces G/H, where the denominator groups arise as the fixed point sets of order-4 automorphisms of some Lie supergroup G. At the Lie algebra level $\mathfrak{g} := \mathrm{Lie}(G)$ we have $(m, n = 0, \ldots, 3)$

$$\mathfrak{g} \;\cong\; \bigoplus_{m=0}^{3} \mathfrak{g}_{(m)}\,, \quad \text{with} \quad \mathfrak{g}_{(0)} \;:=\; \mathrm{Lie}(H) \quad \text{and} \quad [\mathfrak{g}_{(m)}, \mathfrak{g}_{(n)}\} \;\subset\; \mathfrak{g}_{(m+n \bmod 4)}\,. \tag{3.1}$$

Here, $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(2)}$ are generated by bosonic generators while $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(3)}$ by fermionic ones, respectively and $[\cdot, \cdot\}$ denotes the (graded) commutator on $\mathfrak{g}$. As before, we shall also denote $\mathfrak{g}_{(0)}$ by $\mathfrak{h}$.

To write down the superstring action, we consider $g : \Sigma \to G$, where Σ is a worldsheet surface with a Lorentzian signature metric and introduce the current

$$j \;:=\; g^{-1}\mathrm{d}g \;=\; j_{(0)} + j_{(1)} + j_{(2)} + j_{(3)}\,, \quad \text{with} \quad j_{(m)} \;\in\; \mathfrak{g}_{(m)} \tag{3.2}$$

according to the $\mathbb{Z}_4$-decomposition of $\mathfrak{g}$. We again set $A := j_{(0)}$.

The superstring action can be written as a sum of kinetic and Wess-Zumino terms [2, 3, 53],

$$S \;=\; -\frac{T}{2}\int_\Sigma \mathrm{str}\left[j_{(2)} \wedge *j_{(2)} + \kappa j_{(1)} \wedge j_{(3)}\right], \tag{3.3}$$

where $T = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension and 'str' denotes the supertrace on $\mathfrak{g}$ compatible with the $\mathbb{Z}_4$-grading. The κ-symmetry condition requires that $\kappa = \pm 1$; in what follows we shall assume that $\kappa = 1$.[7]

By starting from the Maurer-Cartan equation for the current (3.2)

$$\mathrm{d}j + j \wedge j \;=\; 0 \tag{3.4}$$

[7] The opposite sign choice is related by a parity transformation on Σ.

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and splitting j according to the $\mathbb{Z}_4$-grading of the algebra, we find

$$\begin{aligned} \mathrm{d}A + A\wedge A + j_{(1)}\wedge j_{(3)} + j_{(2)}\wedge j_{(2)} + j_{(3)}\wedge j_{(1)} &= 0\,,\\ \nabla j_{(1)} + j_{(2)}\wedge j_{(3)} + j_{(3)}\wedge j_{(2)} &= 0\,,\\ \nabla j_{(2)} + j_{(1)}\wedge j_{(1)} + j_{(3)}\wedge j_{(3)} &= 0\,,\\ \nabla j_{(3)} + j_{(1)}\wedge j_{(2)} + j_{(2)}\wedge j_{(1)} &= 0\,, \end{aligned} \tag{3.5a}$$

and where we used (2.5). The variation of (3.3) over g together with (3.5a) then yields the following field equations:

$$\begin{aligned} \nabla{*}j_{(2)} + j_{(3)}\wedge j_{(3)} - j_{(1)}\wedge j_{(1)} &= 0\,,\\ j_{(2)}\wedge\big(j_{(1)} + {*}j_{(1)}\big) + \big(j_{(1)} + {*}j_{(1)}\big)\wedge j_{(2)} &= 0\,,\\ j_{(2)}\wedge\big(j_{(3)} - {*}j_{(3)}\big) + \big(j_{(3)} - {*}j_{(3)}\big)\wedge j_{(2)} &= 0\,. \end{aligned} \tag{3.5b}$$

Eqs. (3.5) constitute the full system of the superstring equations in first-order form, i.e. the equations for the algebra-valued one-form j. This system is invariant under the bosonic H-gauge transformations and the fermionic κ-gauge symmetry.[8]

As was shown in [1] for type IIB superstrings on $\mathrm{AdS}_5\times S^5$, the $\mathbb{Z}_4$-grading makes it possible to construct one-parameter families of flat currents which in turn yield infinitely many non-local conserved charges à la Lüscher and Pohlmeyer [4]. In fact, this not only true for the superstring on $\mathrm{AdS}_5\times S^5$ but is a generic feature of models based on cosets with order-4 automorphisms and with an action of the form (3.3).[9] One may verify that the following combination of the components of the current in (3.2)

$$J(\zeta) \;:=\; A + \zeta^{-1} j_{(1)} + \frac{1}{2}(\zeta^2+\zeta^{-2})\, j_{(2)} + \zeta\, j_{(3)} + \frac{1}{2}(\zeta^2-\zeta^{-2})\,{*}j_{(2)}\,, \tag{3.6a}$$

where ζ is a complex spectral parameter, satisfies the flatness condition

$$\mathrm{d}J(\zeta) + J(\zeta)\wedge J(\zeta) \;=\; 0\,, \tag{3.6b}$$

and vice versa, imposing this flatness condition leads to the full system (3.5) of first-order equations for the current j. As before, (3.6b) follows as compatibility condition of an auxiliary linear problem

$$\big[\mathrm{d} + J(\zeta)\big]\psi \;=\; 0\,, \tag{3.6c}$$

where ψ is some G-valued function that depends on the spectral parameter ζ.

3.2 Twistors and generalised self-dual Yang-Mills theory

As we have seen in section 2.2, symmetric space coset models follow upon dimensionally reducing the SDYM equations on $\mathbb{R}^{2,2}$ down to two dimensions. That way two components of the SDYM field $\mathcal{A}_\mu$ combine into a Higgs field leading to the current $j_{(2)}$. Superstrings based on coset superspaces as those mentioned above are described by one-parameter families of flat currents of the form (3.6a). If we want to understand the corresponding superstring equations as a dimensional reduction of some self-duality equations, we in fact need

[8] It is also invariant under $2d$ reparametrisations.

[9] See [30] for the extension to $\mathbb{Z}_k$-graded coset (super)spaces.

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a theory living in eight dimensions since from (3.2)–(3.6) we conclude that we need three Higgs fields that are represented by $j_{(1)}$, $j_{(2)}$ and $j_{(3)}$. Furthermore, like for symmetric coset space models, we should consider the self-duality equations in split signature, i.e. on $\mathbb{R}^{4,4}$. Recall that there are various generalisations of the SDYM equations to (Euclidean) higher dimensions [31–33]. We shall explain which theory leads to the equations (3.5), (3.6) in two dimensions.

Consider complexified eight-dimensional space-time $\mathcal{M}^8 := \mathbb{C}^8$. Furthermore, consider two rank-2 complex vector bundles $\mathcal{S}$ and $\tilde{\mathcal{S}}$ over $\mathcal{M}^8$ and make the identification $T\mathcal{M}^8 \cong \mathcal{S} \otimes \odot^3 \tilde{\mathcal{S}}$, where '$\odot^p$' denotes the p-th symmetric tensor power; see also eqs. (3.18). This will reduce the rotation group $\mathsf{SL}(8,\mathbb{C})$ to $(\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}))/\mathbb{Z}_2$.[10] As before, we may consider the projectivisation of $\tilde{\mathcal{S}}^*$ and introduce $\mathcal{F}^9 := \mathbb{P}(\tilde{\mathcal{S}}^*) \cong \mathbb{C}^8 \times \mathbb{C}P^1$ over $\mathcal{M}^8$. The spaces $\mathcal{M}^8$ and $\mathcal{F}^9$ may be coordinatised by $x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}$ and $(x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}, \lambda_{\dot\alpha})$, where $x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}$ is totally symmetric in its dotted indices and $\lambda_{\dot\alpha}$ are homogeneous coordinates on $\mathbb{C}P^1$. If we introduce $\partial_{\alpha\dot\beta\dot\gamma\dot\delta}$ with[11]

$$\partial_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} x^{\beta\dot\gamma_1\dot\gamma_2\dot\gamma_3} \;=\; \delta_\alpha{}^\beta \delta_{(\dot\beta_1}{}^{\dot\gamma_1} \delta_{\dot\beta_2}{}^{\dot\gamma_2} \delta_{\dot\beta_3)}{}^{\dot\gamma_3}\,, \tag{3.7}$$

where parentheses denote normalised symmetrisation, we then define the twistor distribution to be the rank-2 distribution $\mathcal{D}$ on $\mathcal{F}^9$ given by

$$\mathcal{D} \;:=\; \text{span}\left\{V_\alpha := \lambda^{\dot\beta_1}\lambda^{\dot\beta_2}\lambda^{\dot\beta_3}\partial_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}\right\}\,. \tag{3.8}$$

The reason for this choice of the twistor distribution is that we wish to end up we a Lax *pair* containing the eight components of a gauge potential in eight dimensions; for more details see below.

Since $\mathcal{D}$ is integrable, it defines a foliation of $\mathcal{F}^9$. The resulting quotient will be twistor space, a seven-dimensional complex manifold denoted by $\mathcal{P}^7$,

$$\begin{tikzcd} & \mathcal{F}^9 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{P}^7 & & \mathcal{M}^8 \end{tikzcd} \tag{3.9}$$

where π_2 is the trivial projection and $\pi_1 : (x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}, \lambda_{\dot\alpha}) \mapsto (z^{\alpha\dot\beta_1\dot\beta_2} = x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}\lambda_{\dot\beta_3}, \lambda_{\dot\alpha})$. Hence, $\mathcal{P}^7 \subset \mathbb{C}P^7$ is a holomorphic vector bundle over $\mathbb{C}P^1$ that can best be understood in terms of its global holomorphic sections $H^0(\mathbb{C}P^1, \mathcal{P}^7)$ which are those of $\mathcal{O}(1) \otimes \mathbb{C}^6 \to \mathbb{C}P^1$ with the obvious restrictions on the moduli: $H^0(\mathbb{C}P^1, \mathcal{P}^7) \subset H^0(\mathbb{C}P^1, \mathcal{O}(1) \otimes \mathbb{C}^6)$ with $z^{\alpha\dot\beta_1\dot\beta_2} = x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}\lambda_{\dot\beta_3}$. Notice that $H^1(\mathbb{C}P^1, \mathcal{P}^7) = 0$. Furthermore, a point $x \in \mathcal{M}^8$ corresponds to a projective line $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^7$ in twistor space, while a point $(z,\lambda) \in$

[10] This is somewhat in spirit of para-conformal/quaternionic-conformal manifolds [54, 55] which are $4k$-dimensional complex manifolds $\mathcal{M}$ with the assumption of a factorisation of the tangent bundle $T\mathcal{M}$ into one rank-2 complex vector bundle $\mathcal{S}$ and one rank-$2k$ complex vector bundle $\mathcal{H}$, i.e. $T\mathcal{M} \cong \mathcal{S} \otimes \mathcal{H}$. In this case the rotation group $\mathsf{SL}(4k,\mathbb{C})$ is reduced to $(\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2k,\mathbb{C}))/\mathbb{Z}_2$. In our example, $k = 2$ and we assume in addition that $\mathcal{H}$ is given by $\odot^3\tilde{\mathcal{S}}$ for some rank-2 complex vector bundle $\tilde{\mathcal{S}}$. Hence, we obtain $(\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}))/\mathbb{Z}_2$.

[11] To be more concrete, we have $\partial_{\alpha\dot1\dot1\dot1} := \frac{\partial}{\partial x^{\alpha\dot1\dot1\dot1}}$, $\partial_{\alpha\dot1\dot1\dot2} := \frac{1}{3}\frac{\partial}{\partial x^{\alpha\dot1\dot1\dot2}}$, $\partial_{\alpha\dot1\dot2\dot2} := \frac{1}{3}\frac{\partial}{\partial x^{\alpha\dot1\dot2\dot2}}$ and $\partial_{\alpha\dot2\dot2\dot2} := \frac{\partial}{\partial x^{\alpha\dot2\dot2\dot2}}$.

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$\mathcal{P}^3$ corresponds to a 2-plane inside $\mathcal{M}^8$ that is parametrised by $x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} = x_0^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} + \mu^\alpha\lambda^{\dot\beta_1}\lambda^{\dot\beta_2}\lambda^{\dot\beta_3}$, with $x_0^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} = \text{const.}$ and μ^α arbitrary.

We consider now a rank-$r|s$ holomorphic (super) vector bundle $\mathcal{E} \to \mathcal{P}^7$ and its pull-back $\pi_1^*\mathcal{E} \to \mathcal{F}^9$. Hence, their structure groups are taken to be $\mathsf{GL}(r|s,\mathbb{C})$.[12] Since $\mathcal{P}^7$ and $\mathcal{F}^9$ can be covered by two patches, $\mathcal{U}_\pm$ and $\hat{\mathcal{U}}_\pm$, these bundles are again characterised by transition functions f_{+-}. We shall also assume that $\mathcal{E}$ is topologically trivial and holomorphically trivial when restricted to any $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^7$ for $x \in \mathcal{M}^8$. These conditions then again imply the existence of smooth $\mathsf{GL}(r|s,\mathbb{C})$-valued functions $\psi_\pm$ on $\hat{\mathcal{U}}_\pm$ such that f_{+-} can be decomposed as

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \,, \quad \text{with} \quad \bar\partial_{\mathcal{F}}\psi_\pm \;=\; 0\,. \tag{3.10}$$

Since $V_\alpha^\pm f_{+-} = \lambda_\pm^{\dot\beta_1}\lambda_\pm^{\dot\beta_2}\lambda_\pm^{\dot\beta_3}\partial_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} f_{+-} = 0$, where $V_\alpha^\pm$ are the restrictions of V_α to the coordinate patches $\hat{\mathcal{U}}_\pm$, we find

$$\psi_+ V_\alpha^+ \psi_+^{-1} \;=\; \psi_- V_\alpha^+ \psi_-^{-1} \tag{3.11}$$

on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. Thus, we may introduce a Lie algebra-valued one-form $\mathcal{A}$ on $\mathcal{F}^9$ which has components only along $\mathcal{D}$,

$$V_\alpha \lrcorner \mathcal{A}|_{\hat{\mathcal{U}}_\pm} \;:=\; \mathcal{A}_\alpha^\pm \;=\; \psi_\pm V_\alpha^\pm \psi_\pm^{-1} \;=\; \lambda_\pm^{\dot\beta_1}\lambda_\pm^{\dot\beta_2}\lambda_\pm^{\dot\beta_3}\mathcal{A}_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}\,, \tag{3.12}$$

where $\mathcal{A}_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}$ is $\lambda_\pm$-independent. This can be re-written as

$$(V_\alpha^\pm + \mathcal{A}_\alpha^\pm)\psi_\pm \;=\; \lambda_\pm^{\dot\beta_1}\lambda_\pm^{\dot\beta_2}\lambda_\pm^{\dot\beta_3}\nabla_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}\psi_\pm \;=\; 0\,, \tag{3.13a}$$

with $\nabla_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} := \partial_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3} + \mathcal{A}_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}$. The compatibility conditions are given by

$$\big[\nabla_{\alpha(\dot\beta_1\dot\beta_2\dot\beta_3}, \nabla_{\beta\dot\beta_4\dot\beta_5\dot\beta_6)}\big] \;=\; 0\,, \tag{3.13b}$$

i.e. all dotted indices are symmetrised. Notice that the anti-symmetric tensor product of two vector representations in eight dimensions decomposes under $(\mathsf{SL}(2,\mathbb{C})\times\mathsf{SL}(2,\mathbb{C}))/\mathbb{Z}_2$ as $\mathbf{8}\wedge\mathbf{8} \cong \mathbf{3}\oplus\mathbf{15}\oplus\mathbf{7}\oplus\mathbf{3}$ and the constraints (3.13b) just imply the vanishing of the $\mathbf{7}$-part of the field strength of the gauge potential $\mathcal{A}_{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}$.

In summary, we have a one-to-one correspondence between equivalence classes of holomorphic vector bundles over $\mathcal{P}^7$ that are holomorphically trivial along $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^7$ for $x \in \mathcal{M}^8$ and gauge equivalence classes of solutions to the generalised self-duality equations (3.13b) on $\mathcal{M}^8$. The system (3.13) belongs to the class B_q in Ward's classification scheme [32].

We will now explain that (3.13) yields the Lax connection and the first-order system for the superstring. To this end, we introduce a real structure on $\mathcal{P}^7$ that yields a split signature real slice in $\mathcal{M}^8$. This can be done in a similar way as (2.15). In particular, we consider the involution

$$\tau(z^{\alpha\dot\beta_1\dot\beta_2}, \lambda_{\dot\alpha}) \;:=\; (\bar z^{\beta\dot\gamma_1\dot\gamma_2}C_\beta{}^\alpha C_{\dot\gamma_1}{}^{\dot\beta_1}C_{\dot\gamma_2}{}^{\dot\beta_2}, C_{\dot\alpha}{}^{\dot\beta}\bar\lambda_{\dot\beta})\,, \tag{3.14}$$

[12] We may additionally assume that the Berezinian line bundle $\operatorname{Ber}\mathcal{E}$ is trivial, thus reducing the structure group to $\mathsf{SL}(r|s,\mathbb{C})$.

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where the matrices $C_\alpha{}^\beta$ and $C_{\dot\alpha}{}^{\dot\beta}$ are the same as in (2.15b). We therefore find

$$\tau(x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}) \;=\; \bar x^{\beta\dot\gamma_1\dot\gamma_2\dot\gamma_3} C_\beta{}^\alpha C_{\dot\gamma_1}{}^{\dot\beta_1} C_{\dot\gamma_2}{}^{\dot\beta_2} C_{\dot\gamma_3}{}^{\dot\beta_3}\;. \tag{3.15}$$

as induced involution on $\mathcal{M}^8$. The fixed point set $\tau(x)=x$ is given by

$$x^{1\dot1\dot1\dot1} \;=\; \bar x^{2\dot2\dot2\dot2}\,,\quad x^{1\dot1\dot1\dot2} \;=\; \bar x^{2\dot1\dot2\dot2}\,,\quad x^{1\dot1\dot2\dot2} \;=\; \bar x^{2\dot1\dot1\dot2}\,,\quad x^{1\dot2\dot2\dot2} \;=\; \bar x^{2\dot1\dot1\dot1} \tag{3.16}$$

and defines a split signature space-time $\mathcal{M}^8_\tau \cong \mathbb{R}^{4,4}$:

$$\begin{aligned} \mathrm{d}s^2 \;&=\; \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot\beta_1\dot\gamma_1}\varepsilon_{\dot\beta_2\dot\gamma_2}\varepsilon_{\dot\beta_3\dot\gamma_3}\mathrm{d}x^{\alpha\dot\beta_1\dot\beta_2\dot\beta_3}\mathrm{d}x^{\beta\dot\gamma_1\dot\gamma_2\dot\gamma_3} \\ &=\; |\mathrm{d}x^{1\dot1\dot1\dot1}|^2 - 3|\mathrm{d}x^{1\dot1\dot1\dot2}|^2 + 3|\mathrm{d}x^{1\dot1\dot2\dot2}|^2 - |\mathrm{d}x^{1\dot2\dot2\dot2}|^2\;. \end{aligned} \tag{3.17}$$

We then define

$$\begin{aligned} x^{1\dot1\dot1\dot1} \;&=\; \bar x^{2\dot2\dot2\dot2} \;=:\; \frac{1}{8}[(x^5-3x^8)+\mathrm{i}(3x^1-x^3)]\,,\\ x^{1\dot1\dot1\dot2} \;&=\; \bar x^{2\dot1\dot2\dot2} \;=:\; \frac{1}{8}[(x^6+x^7)+\mathrm{i}(x^2+x^4)]\,,\\ x^{1\dot1\dot2\dot2} \;&=\; \bar x^{2\dot1\dot1\dot2} \;=:\; \frac{1}{8}[(x^5+x^8)-\mathrm{i}(x^1+x^3)]\,,\\ x^{1\dot2\dot2\dot2} \;&=\; \bar x^{2\dot1\dot1\dot1} \;=:\; \frac{1}{8}[(x^6-3x^7)-\mathrm{i}(3x^2-x^4)] \end{aligned} \tag{3.18}$$

for real x^μ with $\mu,\nu,\ldots = 1,\ldots,8$. This parametrisation has been chosen with some hindsight and it will become transparent momentarily. As before, the involution τ can be extended to $\mathcal{E}$ and $\pi_1^*\mathcal{E}$ to end up with real gauge fields taking values in some real form $\mathfrak{g}$ of $\mathfrak{gl}(r|s,\mathbb{C})$.

Inverting (3.18), the linear system (3.13a) on e.g. $\hat{\mathcal{U}}_+$ is given by

$$\begin{aligned} \mathcal{L}_1\psi \;&=\; 0 \;=\; \mathcal{L}_2\psi\,,\\ \mathcal{L}_1 \;&:=\; \lambda^3[(\nabla_5-\nabla_8)-\mathrm{i}(\nabla_1-\nabla_3)] - \lambda^2[(3\nabla_6+\nabla_7)-\mathrm{i}(\nabla_2+3\nabla_4)]\\ &\qquad + \lambda[(3\nabla_5+\nabla_8)+\mathrm{i}(3\nabla_3+\nabla_1)] - [(\nabla_6-\nabla_7)+\mathrm{i}(\nabla_2-\nabla_4)]\,,\\ \mathcal{L}_2 \;&:=\; \lambda^3[(\nabla_6-\nabla_7)-\mathrm{i}(\nabla_2-\nabla_4)] - \lambda^2[(3\nabla_5+\nabla_8)-\mathrm{i}(3\nabla_3+\nabla_1)]\\ &\qquad + \lambda[(3\nabla_6+\nabla_7)+\mathrm{i}(\nabla_2+3\nabla_4)] - [(\nabla_5-\nabla_8)+\mathrm{i}(\nabla_1-\nabla_3)]\,, \end{aligned} \tag{3.19}$$

with $\lambda := \lambda_+$, $\psi := \psi_+$, $\nabla_\mu := \partial_\mu + \mathcal{A}_\mu$ and $\partial_\mu := \partial/\partial x^\mu$. It is a straightforward exercise to compute $[\mathcal{L}_1,\mathcal{L}_2\}$ to arrive at (3.13b) in the coordinates x^μ. We shall postpone presenting the result and first perform an additional transformation of the spectral parameter. In light of our previous discussion, let us again perform the linear fractional transformation (2.22). After some algebraic manipulations, we find that the linear system (3.19) is equivalent to

$$\begin{aligned} \hat{\mathcal{L}}_1\psi \;&=\; 0 \;=\; \hat{\mathcal{L}}_2\psi\,,\\ \hat{\mathcal{L}}_1 \;&:=\; \zeta^3(\nabla_3+\nabla_4)+\zeta^2(\nabla_7+\nabla_8)+\zeta(\nabla_1+\nabla_2)+(\nabla_5+\nabla_6)\,,\\ \hat{\mathcal{L}}_2 \;&:=\; \zeta^3(\nabla_5-\nabla_6)-\zeta^2(\nabla_1-\nabla_2)-\zeta(\nabla_7-\nabla_8)-(\nabla_3-\nabla_4)\;. \end{aligned} \tag{3.20}$$

Of course, both systems (3.19) and (3.20) lead to the same compatibility conditions, though (3.20) looks much simpler and eventually leads us directly to the Lax pair for the superstring. This was the reason for the choice (3.18).

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The compatibility equations are then given by (see also (3.13b))

$$
\begin{aligned}
\mathcal{F}_{12}+\mathcal{F}_{34}+\mathcal{F}_{78}-\mathcal{F}_{56} &= 0\,,\\
\mathcal{F}_{13}-\mathcal{F}_{24}+\mathcal{F}_{67}-\mathcal{F}_{58} &= 0\,,\\
\mathcal{F}_{14}-\mathcal{F}_{23}-\mathcal{F}_{57}+\mathcal{F}_{68} &= 0\,,\\
\mathcal{F}_{15}+\mathcal{F}_{18}-\mathcal{F}_{26}-\mathcal{F}_{27}+\mathcal{F}_{38}-\mathcal{F}_{47} &= 0\,,\\
\mathcal{F}_{16}-\mathcal{F}_{17}-\mathcal{F}_{25}+\mathcal{F}_{28}+\mathcal{F}_{37}-\mathcal{F}_{48} &= 0\,,\\
\mathcal{F}_{35}-\mathcal{F}_{46} &= 0\,,\\
\mathcal{F}_{36}-\mathcal{F}_{45} &= 0\,,
\end{aligned}
\tag{3.21}
$$

where $\mathcal{F}_{\mu\nu} := [\nabla_\mu, \nabla_\nu\}$.

To make contact with our discussion about superstring sigma models, let us assume that $\mathcal{A}_\mu$ depends only on x^1 and x^2 and introduce $\Phi_{1,2} := \mathcal{A}_{3,4}$ and

$$
\begin{aligned}
\Psi_1 &:= \frac{1}{2}(\mathcal{A}_5+\mathcal{A}_6+\mathcal{A}_7-\mathcal{A}_8)\,, &\qquad \Psi_2 &:= \frac{1}{2}(\mathcal{A}_5+\mathcal{A}_6-\mathcal{A}_7+\mathcal{A}_8)\,,\\
\Sigma_1 &:= \frac{1}{2}(-\mathcal{A}_5+\mathcal{A}_6+\mathcal{A}_7+\mathcal{A}_8)\,, &\qquad \Sigma_2 &:= \frac{1}{2}(\mathcal{A}_5-\mathcal{A}_6+\mathcal{A}_7+\mathcal{A}_8)\,.
\end{aligned}
\tag{3.22}
$$

Taking the linear combinations $\frac{1}{2}[\zeta^{-1}\hat{\mathcal{L}}_1 \pm \zeta^{-2}\hat{\mathcal{L}}_2]$, we find from (3.20)

$$
\begin{aligned}
\left[\partial_1+\mathcal{A}_1+\zeta^{-1}\Psi_1+\frac{1}{2}(\zeta^2+\zeta^{-2})\Phi_1+\zeta\Sigma_1+\frac{1}{2}(\zeta^2-\zeta^{-2})\Phi_2\right]\psi &= 0\,,\\
\left[\partial_2+\mathcal{A}_2+\zeta^{-1}\Psi_2+\frac{1}{2}(\zeta^2+\zeta^{-2})\Phi_2+\zeta\Sigma_2+\frac{1}{2}(\zeta^2-\zeta^{-2})\Phi_1\right]\psi &= 0\,.
\end{aligned}
\tag{3.23a}
$$

The compatibility equations of this system are of Hitchin-type

$$
\begin{aligned}
\mathcal{F}_{12}+[\Phi_1,\Phi_2\}+[\Psi_1,\Sigma_2\}+[\Sigma_1,\Psi_2\} &= 0\,,\\
\nabla_1\Psi_2-\nabla_2\Psi_1+[\Phi_1,\Sigma_2\}+[\Sigma_1,\Phi_2\} &= 0\,,\\
\nabla_1\Phi_2-\nabla_2\Phi_1+[\Psi_1,\Psi_2\}+[\Sigma_1,\Sigma_2\} &= 0\,,\\
\nabla_1\Sigma_2-\nabla_2\Sigma_1+[\Phi_1,\Psi_2\}+[\Psi_1,\Phi_2\} &= 0\,,\\
\nabla_1\Phi_1-\nabla_2\Phi_2-[\Psi_1,\Psi_2\}+[\Sigma_1,\Sigma_2\} &= 0\,,\\
[\Phi_1,\Psi_1+\Psi_2\}+[\Psi_1+\Psi_2,\Phi_2\} &= 0\,,\\
[\Phi_1,\Sigma_1-\Sigma_2\}-[\Sigma_1-\Sigma_2,\Phi_2\} &= 0\,,
\end{aligned}
\tag{3.23b}
$$

which, of course, just follow from (3.21) upon assuming that $\mathcal{A}_\mu$ depends only on x^1 and x^2 and using the above definitions of Φ, Ψ and Σ. As before, $\Phi = \Phi_1 \mathrm{d}x^1 + \Phi_2 \mathrm{d}x^2$ and similarly for the others. Eqs. (3.23) represent almost (3.5) and (3.6) in conformal gauge. If we assume that $\mathfrak{g}$ admits a $\mathbb{Z}_4$-grading, then the superstring equations arise as a $\mathbb{Z}_4$-invariant subsector of (3.23). This is analogous to what happened in the symmetric space case. Let $\Omega : \mathfrak{g} \to \mathfrak{g}$ be the $\mathbb{Z}_4$-automorphism of $\mathfrak{g}$. We then may introduce projectors[13] (see also eqs. (3.1); $\mathrm{i} := \sqrt{-1}$)

$$
\mathscr{P}_{(m)} \;:=\; \frac{1}{4}(1+\mathrm{i}^{3m}\Omega+\mathrm{i}^{2m}\Omega^2+\mathrm{i}^m\Omega^3)\,, \quad \text{with} \quad \mathfrak{g}_{(m)} \;=\; \mathscr{P}_{(m)}(\mathfrak{g})\,,
\tag{3.24}
$$

[13]For details on the grading in the case of superstrings on $\mathrm{AdS}_5 \times S^5$, see e.g. [56].

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projecting onto the $\mathfrak{g}_{(m)}$-components of $\mathfrak{g}$. The configurations $(\mathcal{A}, \Psi, \Phi, \Sigma)$ corresponding to the superstring are those which satisfy

$$(\mathcal{A}, \Psi, \Phi, \Sigma) \;=\; (\Omega(\mathcal{A}), \mathrm{i}\Omega(\Psi), -\Omega(\Phi), -\mathrm{i}\Omega(\Sigma))\ . \tag{3.25}$$

Therefore, for such $(\mathcal{A}, \Psi, \Phi, \Sigma)$ we may relable $\mathcal{A} =: A \in \mathfrak{h}$, $\Psi =: j_{(1)} \in \mathfrak{g}_{(1)}$, $\Phi =: j_{(2)} \in \mathfrak{g}_{(2)}$ and $\Sigma =: j_{(3)} \in \mathfrak{g}_{(3)}$ eventually arriving at (3.5) and (3.6).

In summary, the first-order system (3.5) of the superstring based on a coset superspace G/H with the above properties can be obtained as a dimensional reduction of the generalised self-duality type equations (3.13b), (3.21) for a gauge potential $\mathcal{A}_\mu$ with the assumptions $\mathcal{A}_{1,2} \in \mathfrak{h}$, $\mathcal{A}_{3,4} \in \mathfrak{g}_{(2)}$, $\pm\mathcal{A}_5 + \mathcal{A}_6 + \mathcal{A}_7 \mp \mathcal{A}_8 \in \mathfrak{g}_{(1)}$ and $\mathcal{A}_5 \pm \mathcal{A}_6 \mp \mathcal{A}_7 + \mathcal{A}_8 \in \mathfrak{g}_{(3)}$, where $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)} \oplus \mathfrak{g}_{(3)}$. As we have dicussed above, gauge equivalence classes of solutions to the self-duality type equations (3.13b) are in one-to-one correspondence with equivalence classes of holomorphic vector bundles over the twistor space $\mathcal{P}^7$ and its correspondence space $\mathcal{F}^9$ which are subject to certain algebraic constraints. Therefore, all solutions to the superstring equations (3.5) are encoded in these holomorphic vector bundles. Of course, physical solutions to (3.5) should additionally obey the Virasoro constraints putting therefore further assumptions on the admissible vector bundles.

3.3 Remarks

Remark 1. For the sake of concreteness, we only considered a dimensional reduction leading to a Lorentzian worldsheet. Of course, one could perform the reduction differently to end up with a Euclidean worldsheet. Furthermore, as we wanted to re-produce the superstring equations, we considered an anti-holomorphic involution (3.14) on $\mathcal{P}^7$ corresponding to a split signature space-time $\mathbb{R}^{4,4}$. One may instead consider an involution on $\mathcal{P}^7$ leading to an Euclidean signature real slice in $\mathcal{M}^8$ and then try to repeat the above procedure to arrive at a direct generalisation of the Hitchin equations given in [48]. In view of that notice that the Hitchin equations are a key ingredient in recent constructions of strong coupling gluon scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory [23, 24] via the AdS/CFT correspondence (see also [57]). It would be interesting to see whether these generalised equations would play a role when extending the results of [23, 24] to the full background geometry.

Remark 2. Finally, let us make a few comments on non-local charges and hidden symmetry structures. The above twistor description allows for a geometric re-interpretation of the conserved non-local charges for the superstring: In fact, these charges follow from the function ψ appearing in (3.6c) for some particular choice of contour in the worldsheet surface upon expanding it in powers of the spectral parameter [1]. We have just seen how this function is related to the transition functions of the holomorphic vector bundles $\mathcal{E}$ and $\pi_1^*\mathcal{E}$ over the twistor space $\mathcal{P}^7$ and its correspondence space $\mathcal{F}^9$. In this respect, notice that upon reducing (3.19) to two dimensions and making use of the definitions of Ψ, Φ and Σ, one obtains a linear system equivalent to (3.23a). That way one may then study (infinitesimal) deformations of these vector bundles to describe the hidden symmetry algebras for the superstring and furthermore, to give an interpretation of the symmetries in

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terms of sheaf cohomology along the lines presented in the works [44, 58–63]. For example, any deformation algebra of the transition functions of Lie algebra-type can be mapped into a symmetry algebra for the gauge potential (modulo gauge equivalence); see [61, 62] for more details. This twistor re-interpretation may in turn help shed light on the hidden symmetry structures appearing in the gauge theory duals via the holographic correspondence. In view of this it would, for example, be interesting to study the recently uncovered dual (super)conformal symmetry [64–68] (see also [69]) in $\mathcal{N}=4$ SYM theory and superstring theory on $\mathrm{AdS}_5\times S^5$ [57, 70–73] in terms of the twistor approach presented here.[14] Recall that on the string side, the appearance of the dual superconformal symmetry is due to the 'T-self-duality' of the superstring sigma model under a certain combination of bosonic and fermionic T-dualities in Poincaré parametrisation [71, 72]. A way of interpreting this T-duality is then as a dressing transformation on the space of solutions, like a Bäcklund transformation (see, e.g. [75]).

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